

THE DUAL OF THE BOURGAIN–DELBAEN SPACE

BY

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ABSTRACT

It is shown that a \mathcal{L}_∞ -space with separable dual constructed by Bourgain and Delbaen has small Szlenk index and thus does not have a quotient isomorphic to $C(\omega^\omega)$. It follows that this is a \mathcal{L}_∞ -space which is the same size as c_0 in the sense of the Szlenk index but does not contain c_0 . This has some consequences in the theory of uniform homeomorphism of Banach spaces.

1. Introduction

In 1980 Bourgain and Delbaen [BD] published a method of constructing \mathcal{L}_∞ -spaces which produced examples with surprising properties. At the time one of the most interesting aspects of these spaces was that they were the first examples of a separable space with the Radon–Nikodym Property but not isomorphic to a subspace of a separable dual space. In this paper we are not concerned with this property of the examples, but instead with the fact that these \mathcal{L}_∞ -spaces fail to contain c_0 and thus cannot be isomorphic to an isometric $L_1(\mu)$ -predual. (See [JZ].) Such spaces are not well understood and potentially provide a source of interesting examples.

One of our motivations for considering these spaces was that in [JLS] it was shown that a \mathcal{L}_∞ -space with $C(\omega^\omega)$ as a quotient is not uniformly homeomorphic to c_0 . Thus a natural question is whether that means that the only \mathcal{L}_∞ -space which is uniformly homeomorphic to c_0 is c_0 itself. One consequence of the results

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proved here is to show that there is more work to be done by showing that there are \mathcal{L}_∞ -spaces other than c_0 which fail to have $C(\omega^\omega)$ as a quotient.

If the parameters in the construction in [BD] are chosen properly, the dual of the space constructed is separable and therefore by [LS] is isomorphic to ℓ_1 . Our interest is in the w^* -topology on ℓ_1 induced by the example. Because the example does not contain c_0 it is clear that this w^* -topology is much different than that induced by a space such as $C(\alpha)$, $\alpha < \omega_1$, or by a space of affine functions. One difficulty is that because the dual is only isomorphic to ℓ_1 , the standard unit vector basis of ℓ_1 may not be contained in the extreme points of the unit ball. This property of isometric ℓ_1 -preduals is heavily (and often implicitly) used in many analyses of specific \mathcal{L}_∞ -spaces, e.g., [A3], [A4]. Thus some replacement for this approach is necessary. Also the definition of the example is given by constructing embeddings of finite dimensional ℓ_∞ -spaces and thus infinite dimensional information must be extracted from this finite dimensional presentation.

Our approach is to work with the w^* -closure of the ℓ_1 -basis of the dual space as an image of a certain associated compact space with a convenient structure. The w^* -closure of the ℓ_1 -basis, C , is large enough to contain most of the important information about the dual, since $D = \overline{\text{co}} \pm C^{\|\cdot\|}$ will contain a multiple of the unit ball. On the other hand, we do not have good information about the extreme points and the w^* -topology of this set D . To overcome this problem we create this associated compact space and we work through the Choquet theorem and use special information about C which is encoded in the associated compact space.

In the next section we will recall the definition of the example as given in [BD] and we will show that the natural coordinate functionals are a basis for the dual and are equivalent to the usual unit vector basis of ℓ_1 . In Section 3 we develop an approach to computing the Szlenk index which allows us to move from information about a subset of the dual to its signed convex hull. This approach may be useful for estimating the Szlenk index in other situations and thus we develop the ideas in a fairly general setting. As part of this we introduce a notion of integration for ordinal-valued functions of a real variable. In Section 4 we estimate the Szlenk index for each $\epsilon > 0$. In the last section we discuss some possible extensions of the method of construction given by Bourgain and Delbaen.

We use standard notation and terminology from Banach space theory as may be found in the books [LTI] and [LTII]. We consider only Banach spaces over the real numbers although much can be adapted to the complex case. In Section 3 we will need the Szlenk index, [Szl], so we recall the definition here.

Definition 1.1: Let X be a Banach space and let $A \subset X$ and let $B \subset X^*$. Given $\epsilon > 0$ we define a family of subsets of B indexed by the ordinals less than or equal to ω_1 .

Let $P_0(\epsilon, A, B) = B$. If $P_\alpha(\epsilon, A, B)$ has been defined, let

$$(1.1) \quad P_{\alpha+1}(\epsilon, A, B) = \{b \in B : \text{there exist } (a_n) \subset A, (b_n) \subset P_\alpha(\epsilon, A, B) \\ \text{such that } w^* \lim b_n = b, \lim b_n(a_n) \geq \epsilon, w \lim a_n = 0\}.$$

If α is a limit ordinal,

$$P_\alpha(\epsilon, A, B) = \bigcap_{\beta < \alpha} P_\beta(\epsilon, A, B).$$

Let $\eta(\epsilon, A, B)$ be the smallest ordinal α such that $P_\alpha(\epsilon, A, B) = \emptyset$.

Usually B is a w^* -closed subset of B_{X^*} and A is B_X , where B_{X^*} and B_X are the unit balls of X^* and X , respectively. If X^* is separable, then $\eta(\epsilon, A, B)$ is defined and countable. Otherwise the convention is to define $\eta(\epsilon, A, B) = \omega_1$ if there is no countable ordinal for which the set $P_\alpha(\epsilon, A, B)$ is empty. In this paper we will always assume that $A = B_X$ so we will omit this from the notation and write $P_\alpha(\epsilon, B)$.

In the case $A = B_X$ and X^* separable it is often convenient to use a different definition of the Szlenk index, which yields a slightly different dependence on ϵ , but in most applications gives equivalent results. In this case the definition of $P_{\alpha+1}(\epsilon, A, B)$ is replaced by

$$(1.2) \quad P_{\alpha+1}(\epsilon, A, B) = \{b \in B : \text{there exists } (b_n) \subset P_\alpha(\epsilon, A, B) \\ \text{such that } w^* \lim b_n = b, \text{ and for all } n \neq m, \|b_n - b_m\| \geq \epsilon\}.$$

We will refer to this second version of the Szlenk index as the modified Szlenk index.

2. The Bourgain-Delbaen spaces

In this section we describe the construction of \mathcal{L}_∞ -spaces due to Bourgain and Delbaen. We will depart slightly from their notation and construction, but this is only a matter of convenience. The approach is to build a subspace of ℓ_∞ by defining a family of consistent embeddings of $\ell_\infty^{d_n}$ into ℓ_∞ , where (d_n) is some sequence of integers tending to infinity rapidly. The sequence (d_n) is defined inductively as are the embeddings.

Fix two positive real numbers a, b and a number $\lambda > 1$ such that $b < a \leq 1$ and $a + 2b\lambda < \lambda$. We define $d_1 = 1, d_2 = 2$ and assume that d_k has been defined

for $k = 1, 2, \dots, n$. We define $d_{n+1} - d_n$ to be the cardinality of the set of tuples $(\sigma', i, m, \sigma'', j)$ such that $1 \leq m < n$, $1 \leq i \leq d_m$, $1 \leq j \leq d_n$ and σ' and σ'' are 1 or -1 . By enumerating the set of tuples by the integers k , $d_n < k \leq d_{n+1}$, we can inductively define a map ϕ from $\mathbb{N} \setminus \{1, 2\}$ to the set of such tuples, $(\sigma', i, m, \sigma'', j)$.

For each $k \in \mathbb{N}$, let e_k^* denote the k -th coordinate functional of ℓ_∞ , and e_k the k -th coordinate element, i.e., the element of ℓ_∞ which is 0 at each coordinate except the k -th and 1 in the k -th. To define the embeddings, let $E_n = [e_k : k \leq d_n]$ for each n and define for $m < n$ inductively $i_{m,n}: E_m \rightarrow E_n$ as follows. We define $i_{1,2}(te_1) = te_1 = e_1^*(te_1)e_1$ for all t and suppose that $i_{m,n}$ has been defined for all $m < n$. To define an extension map from E_n into E_{n+1} for each k , $d_n < k \leq d_{n+1}$, we define a functional $f_{\phi(k)} \in E_n^*$ by

$$f_{\phi(k)}(x) = a\sigma'e_i^*(x) + b\sigma''e_j^*(x - i_{m,n}\pi_mx),$$

where $\pi_m: \ell_\infty \rightarrow E_m$ is standard projection and $\phi(k) = (\sigma', i, m, \sigma'', j)$. Then

$$i_{n,n+1}(x) = x + \sum_{k=d_n+1}^{d_{n+1}} f_{\phi(k)}(x)e_k$$

for all $x \in E_n$. Using this map we can define $i_{m,n+1}(x) = i_{n,n+1}(i_{m,n}(x))$ for all $m < n$ and $x \in E_m$. In [BD] it is shown that $\|i_{m,n}\| \leq \lambda$ for all $m < n$, and thus considering ℓ_∞ as the dual of ℓ_1 , the w^* -operator limit P_m of $(i_{m,n}\pi_m)_{n=m+1}^\infty$ exists for each m . ($P_m(x)$ is just the coordinate-wise limit of $i_{m,n}(x)$ for each x and each coordinate is eventually constant.) Notice that we can now replace the definition of $f_{\phi(k)}$ by

$$f_{\phi(k)}(x) = a\sigma'e_i^*(x - P_0x) + b\sigma''e_j^*(x - P_mx),$$

where $P_0 = 0$. Rewriting this in dual form we have

$$f_{\phi(k)}(x) = a\sigma'(I - P_0^*)e_i^*(x) + b\sigma''(I - P_m^*)e_j^*(x).$$

We are interested in the spaces $X_{a,b} = [P_m(E_m) : m \in \mathbb{N}]$, where a, b are fixed constants as above. It follows easily that $X_{a,b}$ is a \mathcal{L}_∞ -space and in [BD] some of the Banach space properties of these spaces are determined. If $a = 1$ the dual of $X_{a,b}$ is non-separable and thus is not of interest to us here. **Thus we assume that $a < 1$ unless otherwise noted.** We will also suppress the subscripts a, b from now on.

Our first task is to show that the dual of X is isomorphic to ℓ_1 in a very concrete sense. Notice that for each m , P_m can be considered either as a map

from ℓ_∞ into X or as a map from X into itself. Thus the range of P_m^* is contained in $[e_k^* : k \leq d_m]$, either in ℓ_∞^* or by restriction to X , as elements of X^* .

PROPOSITION 2.1: *Let Q be the quotient map from ℓ_∞^* onto X^* . Then $(Q(e_n^*))$ is equivalent to the standard unit vector basis of ℓ_1 and $Q[e_n^* : n \in \mathbb{N}] = X^*$.*

Proof: Because $\|P_m\| \leq \lambda$ and for $g \in [e_1, e_2, \dots, e_{d_m}]$ and $k \leq d_m$,

$$P_m^*Q(e_k^*)(g) = e_k^*(P_m(g)) = e_k^*(g),$$

for each m , it follows that

$$\|Q \sum_{k=1}^{d_m} a_k e_k^*\|_{X^*} \geq \lambda^{-1} \|P_m^*Q(\sum_{k=1}^{d_m} a_k e_k^*)\|_{\ell_\infty^*} = \lambda^{-1} \|\sum_{k=1}^{d_m} a_k e_k^*\|_{\ell_\infty^*}.$$

This proves the first assertion.

For the second we will show that the w^* -closure of $\{Q(e_n^*) : n \in \mathbb{N}\}$ is contained in $[Q(e_n^*) : n \in \mathbb{N}]$. It then follows from the Choquet theorem and Smulian's theorem that $[Q(e_n^*) : n \in \mathbb{N}]$ is w^* -closed and hence equal to X^* . (See [A4], Lemma 1.)

Let x^* be a w^* -limit point of $(Qe_k^*)_{k \in M}$, for some infinite subset M of \mathbb{N} . We may assume that $\lim_{k \in M} Qe_k^*(P_m(e_r)) = x^*(P_m(e_r))$ for each $r \leq d_m$ and each m . Let $\phi(k) = (\sigma'_k, i_k, m_k, \sigma''_k, j_k)$. We may also assume, by passing to a smaller index set if necessary, that $\sigma'_k = \sigma'$ and $\sigma''_k = \sigma''$ for all $k \in M$. Consider (m_k) . If $\sup m_k = \infty$, then $b\sigma''(I - P_{m_k}^*)e_{j_k}^*(x) = 0$ for all $x \in P_m(E_m)$ for $m \leq m_k$ and thus any w^* -limit point of $(e_k^*)_{k \in M}$ is a w^* -limit point of $(a\sigma'(I - P_0^*)e_{i_k}^*)$. If $\sup m_k = m < \infty$, then $i_k \leq d_m$ and $(a\sigma'(I - P_0^*)e_{i_k}^*)$ has a constant subsequence. Thus any w^* -limit point of $(e_k^*)_{k \in M}$ is of the form $a\sigma'(I - P_0^*)e_{i_k}^* + y^*$ where y^* is a w^* -limit point of $(b\sigma''(I - P_{m_k}^*)e_{j_k}^*)$. Notice that in both cases we have replaced looking for a w^* -limit of $(e_k^*) = ((I - P_0^*)e_k^*)$ by looking for a w^* -limit of $(c(I - P_{m_k}^*)e_{r_k}^*)$ where $|c| = a$ or b . Therefore we can find a convergent (absolutely summable) series of terms of the form $c_j(I - P_{m_j}^*)e_{r_k}^*$, $|c_j| \leq a^{j-1}$, with limit x^* . Actually $c_j = \pm a^s b^{j-s}$ for some s , $0 \leq s \leq j$, and $c_{j+1} = \pm ac_j$ or $c_{j+1} = \pm bc_j$. Because $(I - P_m^*)(e_k^*) \in [e_j^* : j \in \mathbb{N}]$, for all m, k , it follows that $x^* \in [e_j^* : j \in \mathbb{N}]$. ■

Remark 2.2: In [GKL,GKL1] it is shown that a Banach space which is uniformly homeomorphic to c_0 must have Szlenk index which behaves as the Szlenk index of c_0 . It may be possible to use the representation of the w^* -closure of the ℓ_1 -basis contained in the previous proof to get a lower estimate on the Szlenk index and

thereby show that the Bourgain–Delbaen space is not uniformly homeomorphic to c_0 .

3. Estimating ordinal indices

We begin by considering an abstract system of derived sets of a metric space. Eventually we will consider the specific cases where this is the usual topological derived sets or the Szlenk sets.

Definition 3.1: Let K be a closed subset of a topological space (X, τ) and let $d(\cdot, \cdot)$ be a metric on X (which may not be compatible with the topology τ). A **δ -system of derived sets** is a family, $(K^{(\alpha)})_{\alpha < \omega_1}$, of closed subsets of K such that

- (1) there exists some ordinal $\beta_0 < \omega_1$ such that $K^{(\alpha)} = \emptyset$ if $\alpha > \beta_0$,
- (2) if $\alpha < \beta$, then $K^{(\alpha)} \supset K^{(\beta)}$,
- (3) if β is a limit ordinal, $\bigcap_{\alpha < \beta} K^{(\alpha)} = K^{(\beta)}$.
- (4) if $x_n \in K^{(\alpha)}$ for all $n \in \mathbb{N}$ and $d(x_n, x_m) \geq \delta$ for all $n \neq m, n, m \in \mathbb{N}$ and $\tau - \lim x_n = x$, then $x \in K^{(\alpha+1)}$.

For each $\alpha < \omega_1$ let $K^{d(\alpha)} = K^{(\alpha)} \setminus K^{(\alpha+1)}$.

We are interested in determining how a Szlenk-like index of a set of finite positive measures on K as elements of $C(K)^*$ behaves with respect to this derivation on K . To measure this we introduce for each $\epsilon > 0$ and finite measure μ on K the ϵ -distribution function of μ , $f_{\epsilon, \mu}$, from $(0, \infty)$ into $[0, \omega_1)$ but with support in $(0, \epsilon]$.

To understand the approach consider the following problem. Suppose that g is a nice function on $(0, \infty)$ with values in the countable ordinals. Is there a sensible notion of area under the graph of g ?

Because it is not at all clear how to multiply real numbers and ordinals, let's take a discrete approach. Fix $\epsilon > 0$. For an indicator function $\gamma 1_{(0, n\epsilon)}$ where $n \in \mathbb{N}$, we want the ϵ -area to be $\gamma \cdot n$. Given an ordinal valued function g on $(0, \infty)$ the ϵ -area under g should be the supremum of the ordinal sums $\gamma_1 + \dots + \gamma_k$ of ϵ -areas of disjoint ϵ -“rectangles” of width ϵ and height $\gamma_i, i = 1, 2, \dots, k$, which fit under the graph of g . By a “rectangle” we mean a set of the form $A \times B$ where A is Lebesgue measurable and B is an interval. There is another difficulty in this in that the non-commutativity of the addition makes this sensitive to the order in which the rectangles are taken. To control this difficulty we need the order of the addition of the rectangles to reflect the values of the function g . To deal with this we use a geometric approach. We think of the ordinal

sum $\gamma_1 + \dots + \gamma_k$ as the value of a new function g' on $(0, \epsilon]$ with ϵ -area under g' approximating the ϵ -area under g . To be an admissible approximation we require that for each x the segments in the rectangles above x be in an order which respects the order of the corresponding segments under the graph of g . More precisely, there is an injective function ψ from $\{(x, y) : 0 < x \leq \epsilon, 0 \leq y \leq g'(x)\}$ into $\{(x, y) : 0 < x, 0 \leq y \leq g(x)\}$ such that if for some x , $\psi(x, y_1) = (s, t_1)$ and $\psi(x, y_2) = (s, t_2)$, then $t_1 < t_2$ implies $y_1 < y_2$. Thus the region under g' is the image under an order preserving (in the second coordinate only) rearrangement of a portion of the region under g .

At first it may seem that we have drifted far from the original problem. The connection to our problem is that intuitively the ϵ -Szlenk index does something similar to computing the ϵ -area under the distribution of a measure. Before we introduce precise formulations, consider the measure

$$\mu = \frac{3}{4}\delta_\omega + \frac{1}{4}\delta_{\omega\omega}$$

in the dual of $C(\omega^\omega)$ and its position in the Szlenk sets of the ball of $C(\omega^\omega)^*$. Notice that if $1/2 < \epsilon \leq 3/4$, μ is in $P_1(\epsilon)$ but no higher Szlenk set. If $1/4 < \epsilon \leq 1/2$, μ is in $P_2(\epsilon)$, and if $\epsilon \leq 1/4$, μ is in $P_{\omega+3}(\epsilon)$. Now consider the distribution function

$$g(t) = \omega 1_{(0, 1/4]} + 1_{(1/4, 1]}$$

and notice that the ϵ -area we have loosely defined above is the same as the ϵ -Szlenk index of μ , i.e., the $3/4$ -area is 1, the $1/2$ -area is 2 and the $1/4$ -area is $\omega + 3$.

Now we will begin making these ideas precise. The definition of the ϵ -distribution function is via an inductive procedure. We will define a sequence of functions, g_1, g_2, \dots, g_n from $(0, \infty)$ into $[0, \omega_1)$, and a non-increasing sequence of ordinals $\gamma_1, \dots, \gamma_n$; then $f_{\epsilon, \mu}(t)$ will be $\sum_{i=1}^n \gamma_i + g_n(t)$ for some n and all $t \leq \epsilon$.

First we assume that $\mu(K^{d(\alpha)}) \neq 0$ for only finitely many α . Let $\alpha_1 > \alpha_2 > \dots > \alpha_k$ be the finite sequence of ordinals such that $\lambda_i = \mu(K^{d(\alpha_i)}) > 0$ for each i and $\mu(K) = \sum_{i=1}^k \lambda_i$ and define $g_1(t) = \alpha_i$ if $\sum_{j=1}^{i-1} \lambda_j < t \leq \sum_{j=1}^i \lambda_j$, and $g_1(t) = 0$ for $t > \sum_{i=1}^k \lambda_i$.

Before giving a formal description of the inductive procedure, let us consider the following intuitive idea for a constructive approach to finding the ϵ -area. Notice that the graph of g_1 is decreasing. We would like to take the largest ordinal β such that $g_1(\epsilon) \geq \beta$, i.e., $g_1(\epsilon)$, let $\gamma_1 = \beta$ and define a new function g_2 as the decreasing rearrangement of $g_1 - 1_{(0, \epsilon]} \gamma_1$. The rectangle of width ϵ and height γ_1 is our first approximation to the area under g_1 and the region under

g_2 is the remainder. Next, we would apply the procedure to g_2 to get a new ordinal $\gamma_2 = g_2(\epsilon)$ and let g_3 be the decreasing rearrangement of $g_2 - 1_{(0,\epsilon]}\gamma_2$. Proceeding inductively, we would find (g_i) and (γ_i) . Notice that $\gamma_i \geq \gamma_{i+1}$ and for only finitely many i can we have equality. Thus at some stage $\gamma_n = 0$ and the procedure produces nothing new.

Because of some features of ordinal addition, it turns out that this procedure may produce a little smaller function than we would like. To avoid this it is necessary to require that $\gamma_i = \omega^{\beta_i}$ for some β_i and thus γ_i may be strictly smaller than $g_i(\epsilon)$. In the formal procedure below we will also describe in detail a method for obtaining the decreasing rearrangement which will allow us to extract some additional information for use later. The main step in the procedure is contained in the following lemma. Recall that if γ and β are ordinals such that $\beta < \gamma$ then $\gamma - \beta$ is the ordinal ρ such that $\beta + \rho = \gamma$. (See [H], page 74.) In the statement of the lemma and below λ denotes Lebesgue measure.

LEMMA 3.2: *Suppose that g and h are left continuous non-increasing functions from $(0, \infty)$ into $[0, \omega_1)$ such that there exists $A < \infty$ with $g(t) = 0 = h(t)$ for all $t > A$, $g(t) \leq h(t)$ for all t , and the range of each is a finite set of ordinals. Let $I = (a, b]$ be an interval on which g and h are constant and let $\gamma \leq g(t)$ for $t \in I$. Then if G and H are the non-increasing left-continuous rearrangements of $g - \gamma 1_I$ and $h - \gamma 1_I$, respectively, then $G(t) \leq H(t)$ for all t and $\lambda(\{t : g(t) + 1 \leq h(t)\}) \leq \lambda(\{t : G(t) + 1 \leq H(t)\})$.*

Proof: Let $s = \sup\{t : g(b) - \gamma < g(t)\}$ and $r = \sup\{t : h(b) - \gamma < h(t)\}$. Because g and h are non-increasing,

$$G(t) = \begin{cases} g(t) & \text{if } t \leq a \text{ or } t > s \\ g(t + (b - a)) & \text{if } a < t \leq s - (b - a) \\ g(t - (s - b)) - \gamma & \text{if } s - (b - a) < t \leq s \end{cases}$$

and

$$H(t) = \begin{cases} h(t) & \text{if } t \leq a \text{ or } t > r \\ h(t + (b - a)) & \text{if } a < t \leq r - (b - a) \\ h(t - (r - b)) - \gamma & \text{if } r - (b - a) < t \leq r. \end{cases}$$

Observe that $G(t) \leq H(t)$ for all $t \leq p = \min(r, s) - (b - a)$ and

$$\lambda(\{t \leq p : g(t) + 1 \leq h(t)\}) = \lambda(\{t \leq p : G(t) + 1 \leq H(t)\}).$$

Similarly, if $q = \max(r, s)$,

$$\lambda(\{t > q : g(t) + 1 \leq h(t)\}) = \lambda(\{t > q : G(t) + 1 \leq H(t)\}).$$

To see that $G(t) \leq H(t)$ for $q \geq t > p$ we note that if we do the same rearrangement of $g - \gamma 1_I$ as for obtaining H from $h - \gamma 1_I$, we get

$$G_1(t) = \begin{cases} g(t) & \text{if } t \leq a \text{ or } t > r \\ g(t + (b - a)) & \text{if } a < t \leq r - (b - a) \\ g(t - (r - b)) - \gamma & \text{if } r - (b - a) < t \leq r \end{cases}$$

and clearly all of the conclusions hold for G_1 in place of G . Now if G_1 is not non-increasing, we have two cases to consider. If $r < s$, then $G(t) = G_1(t - (s - r)) \leq G_1(t) \leq H(t)$ for $s - (b - a) < t \leq s$ and $G(t) = G_1(t + (b - a)) \leq G_1(t) \leq H(t)$ for $r - (b - a) < t \leq s - (b - a)$. Also

$$\begin{aligned} & \{t : G(t) + 1 \leq H(t)\} \\ & \supset \{t : G_1(t) + 1 \leq H(t), r - (b - a) < t \leq s - (b - a)\} \cup (s - (b - a), s]. \end{aligned}$$

Thus the conclusion holds in this case.

If $r > s$, then $G(t) = G_1(t + (r - s)) \leq H(t + (r - s)) \leq H(t)$ for $s - (b - a) < t \leq s$, and $G(t) = G_1(t - (b - a)) \leq G_1(t) \leq H(t)$ for $s < t \leq r$. In this case

$$\begin{aligned} & \{t : G(t) + 1 \leq H(t)\} \\ & \supset \{t : G_1(t + (r - s)) + 1 \leq H(t + (r - s)), s - (b - a) < t \leq s\} \cup (s, r] \end{aligned}$$

and the conclusion holds here too. ■

The next lemma follows from a finite number of applications of Lemma 3.2.

LEMMA 3.3: *Suppose that g and h are left continuous non-increasing functions from $(0, \infty)$ into $[0, \omega_1)$ such that $g(t) = 0 = h(t)$ for all $t > A$ for some A , $g(t) \leq h(t)$ for all t , and the range of each is a finite set of ordinals. Let $\epsilon > 0$ and $\gamma > 0$ such that $\gamma \leq g(\epsilon)$. Then if G and H are the non-increasing rearrangements of $g - \gamma 1_{(0, \epsilon]}$ and $h - \gamma 1_{(0, \epsilon]}$, respectively, then $G(t) \leq H(t)$ for all t and*

$$\lambda(\{t : g(t) + 1 \leq h(t)\}) \leq \lambda(\{t : G(t) + 1 \leq H(t)\}).$$

Proof: There are a finite number of disjoint, left-open, right closed intervals $I_j, j = 1, 2, \dots, J$, such that $1_{(0, \epsilon]}$, g and h are constant on each, and $\bigcup_{j=1}^J I_j = \text{supp } h$. We may assume that the intervals are ordered so that if $j_1 < j_2$, $s \in I_{j_1}$, and $t \in I_{j_2}$, then $s > t$. There is some smallest index j_0 such that $I_{j_0} \subset (0, \epsilon]$. Applying Lemma 3.2 to I_{j_0} , g and h , we get rearrangements $g^{(1)}$ and $h^{(1)}$ of $g - \gamma 1_{I_{j_0}}$ and $h - \gamma 1_{I_{j_0}}$, respectively. Next we repeat the process with I_{j_0+1} , $g^{(1)}$ and $h^{(1)}$ to obtain $g^{(2)}$ and $h^{(2)}$. Clearly, this process produces the required non-increasing rearrangements of $g - \gamma 1_{(0, \epsilon]}$ and $h - \gamma 1_{(0, \epsilon]}$ at stage $J - j_0 + 1$. Because $g^{(j)} \leq h^{(j)}$ and

$$\lambda(\{t : g^{(j)}(t) + 1 \leq h^{(j)}(t)\}) \leq \lambda(\{t : g^{(j+1)}(t) + 1 \leq h^{(j+1)}(t)\})$$

for each j , the required properties follow immediately ■

Remark 3.4: Notice that if h is non-increasing as in Lemma 3.3 and ρ is an ordinal such that $\rho \cdot \omega < h(\epsilon)$, then $h - \rho 1_{(0,\epsilon]} = h$. Thus if too small an ordinal is chosen, there is no effect.

The next proposition will enable us to define the ϵ -distribution. Below we use summations of ordinals with the understanding that $\sum_{i=1}^n \gamma_i = \gamma_1 + \gamma_2 + \dots + \gamma_n$ in that order.

PROPOSITION 3.5: *Let $\epsilon > 0$ and let $g_0 : (0, \infty) \rightarrow [0, \omega_1)$ be a left continuous, non-increasing function with range a finite set such that for some $t_0 < \infty$, $g_0(t) = 0$ for all $t > t_0$. Then there exists a finite sequence of left continuous, non-increasing functions $(g_i)_{i=1}^n$ from $(0, \infty)$ into $[0, \omega_1)$ and a non-increasing sequence of ordinals $(\gamma_i)_{i=0}^{n-1}$ such that for each $i < n$ and $\alpha < \omega_1$,*

$$\lambda(\{t : g_{i+1}(t) = \alpha\}) = \lambda(\{t : g_i(t) - \gamma_i 1_{(0,\epsilon]}(t) = \alpha\}),$$

i.e., g_{i+1} is a decreasing rearrangement of $g_i - \gamma_i 1_{(0,\epsilon]}$, $\gamma_i = \omega^{\beta_i}$ for some β_i , and $g_n(t) = 0$, for all $t \geq \epsilon$.

Moreover, if g_0 and h_0 are two non-increasing functions as above, $g_0(t) \leq h_0(t)$ for all t , and $(g_i)_{i=1}^n$, $(\gamma_i)_{i=1}^{n-1}$, and $(h_i)_{i=1}^m$, $(\eta_i)_{i=1}^{m-1}$, are the corresponding sequences of functions and ordinals produced, then $\sum_{i=1}^{n-1} \gamma_i + g_n(t) \leq \sum_{i=1}^{m-1} \eta_i + h_m(t)$, for all $t \leq \epsilon$. Further, if $\lambda\{t : g_0(t) + 1 \leq h_0(t)\} \geq \epsilon$, then $\sum_{i=1}^{n-1} \gamma_i + g_n(\epsilon) + 1 \leq h_m(\epsilon)$.

Proof: The proof proceeds by constructing inductively the sequence (g_i) . In order to prove the moreover assertion we will work with h_0 at the same time and produce the corresponding sequence (h_i) .

Suppose that we have g_i and h_i , $1, 2, \dots, k$, such that $g_i \leq h_i$ for each i . If $g_k(\epsilon) = 0$, the construction of the sequence (g_i) is complete. If not let β_k be the largest ordinal β such that $\omega^\beta \leq g_k(\epsilon)$. Let $g = g_k$, $h = h_k$, $\gamma = \omega^{\beta_k}$, and $I = (0, \epsilon]$. Applying Lemma 3.3 we let $g_{k+1} = G$ and $h_{k+1} = H$ be the decreasing rearrangements of $g_k - \gamma_k 1_I$ and $h_k - \gamma_k 1_I$ such that $G \leq H$. Moreover, $\lambda(\{t : g_k(t) + 1 \leq h_k(t)\}) \leq \lambda(\{t : g_{k+1}(t) + 1 \leq h_{k+1}(t)\})$.

Notice that if $h_k(\epsilon) \geq \gamma_k \cdot \omega$, $h_k - \gamma_k 1_I = h_k$. Thus if this occurs for some k , $h_k = h_i$ for all i , $k \leq i \leq n$, and

$$\sum_j^{k-1} \eta_j + h_k(\epsilon) > \sum_j^{i-1} \gamma_j + g_i(\epsilon) + 1$$

for each i . If $h_k(\epsilon) < \gamma_k \cdot \omega$, for all $k \leq n - 1$, then each step of the construction of (g_i) is also a step in the construction of (h_i) with $\eta_k = \gamma_k$. Clearly,

$$\sum_{j=n}^{i-1} \eta_j + h_i \geq g_n$$

for $i = n, n + 1, \dots, m$. This completes the proof of all of the conclusions except for the final assertion in the case $h_k(\epsilon) < \gamma_k \cdot \omega$.

Because $\lambda(\{t : g_n(t) + 1 \leq h_n(t)\}) \geq \epsilon$, at step n either $h_n(t) = 0$ for all $t > \epsilon$ and $h_n(t) \geq g_n(t) + 1$ for all $t, 0 \leq t \leq \epsilon$, or $h_n(t) > 0$ for some $t > \epsilon$. The first case satisfies the conclusion of the proposition. In the second case observe that for each $i, \sum_{j=1}^i \gamma_j + h_{i+1}(\epsilon) \geq \sum_{j=1}^{i-1} \gamma_j + h_i(\epsilon)$. Because $h_n(t) > 0$ for some $t > \epsilon$, it follows that there is a largest $\eta_n = \omega^{\beta_n} > 0$ such that $\eta_n \leq h_n(\epsilon)$. Because $\eta_n > g_n(\epsilon)$, the proof is complete. ■

We now introduce terminology for some of the ingredients of Proposition 3.5 and its proof.

Definition 3.6: Suppose g is a non-increasing left-continuous function from $(0, \infty)$ into $(0, \omega_1)$ and $\epsilon > 0$. If $\gamma \leq g(\epsilon)$ and f is the decreasing rearrangement of $g - \gamma 1_{(0, \epsilon]}$ then $h = \gamma 1_{(0, \epsilon]} + f$ will be said to be an ϵ -**compression** of g (by γ).

Let

$$C(g, \epsilon) = \sup\{H(\epsilon) : \text{there exist non-increasing left-continuous simple functions } (h_i)_{i=1}^n, h_1 \leq g, h_{i+1} \text{ is an } \epsilon\text{-compression of } h_i, H = h_n\}.$$

For a positive finite measure μ on K let $g(t) = \sup\{\alpha : \mu(K^{(\alpha)}) \geq t\}$ for all $t > 0$ and define $C(\mu, \epsilon) = C(g, \epsilon)$. (We let the supremum of an empty set of ordinals be 0.) We will call g the **derived height** of μ and $C(g, \epsilon)$ the ϵ -**area** under g .

It is not hard to see that the procedure used in the proof of Proposition 3.5 will produce the value of $C(g, \epsilon)$ if g is simple. In that case with g_j and γ_j as in the proof we let $h_i = \sum_{j=1}^{i-1} \gamma_j 1_{(0, \epsilon]} + g_i$. It is important in achieving the supremum that for each j, γ_j is of the form ω^{β_j} . This avoids lowering the sum by taking the wrong order, e.g., $\omega^2 + 1$ and ω sum (in that order) to $\omega^2 + \omega$ but ω^2, ω , and 1 sum to $\omega^2 + \omega + 1$.

Observe that for a measure μ as in Definition 3.6, if for some $t, g(t) = \alpha, \mu(K^{(\alpha)}) \geq t$. Also, if (t_n) is an increasing sequence of positive numbers with limit t and $g(t_n) = \alpha_n$ for each $n, (\alpha_n)$ must eventually be constant.

Thus $\mu(\bigcup K^{(\alpha_n)}) = \lim \mu(K^{(\alpha_n)}) \geq \lim t_n$ and $g(t) = \lim g(t_n)$, i.e., g is left-continuous.

Also notice that if g and h are non-increasing functions as in the statement of the proposition but not necessarily simple, then the final conclusion of Proposition 3.5 still holds, i.e, $C(g, \epsilon) + 1 \leq C(h, \epsilon)$. Indeed, if g_1 is a simple function with $g_1 \leq g$ and A is the set where $h(t) \geq g(t) + 1$, then $g_1 + 1_A \leq h$. It follows easily that there is a non-increasing simple function h_1 such that $g_1 + 1_A \leq h_1 \leq h$. Thus $C(g_1, \epsilon) + 1 \leq C(h_1, \epsilon) \leq C(h, \epsilon)$. Taking the supremum over all such g_1 gives the result.

Example 3.7: If we return to our previous example

$$g(t) = \omega 1_{(0,1/4]} + 1_{(1/4,1]}$$

and let $\epsilon = 1/2$, then $C(g, 1/2) = 2$ because $h_1 = g$ and

$$H = h_2 = \omega 1_{(0,1/4]} + 2 \cdot 1_{(1/4,1/2]}.$$

If $\epsilon = 1/4$ then $C(g, 1/4) = \omega + 3$. Indeed, $h_1 = g$,

$$\begin{aligned} h_2 &= (\omega + 1)1_{(0,1/4]} + 1_{(1/4,3/4]}, \\ h_3 &= (\omega + 2)1_{(0,1/4]} + 1_{(1/4,1/2]}, \\ H = h_4 &= (\omega + 3)1_{(0,1/4]}. \end{aligned}$$

Remark 3.8: The definition of ϵ -area can be adapted to accommodate different values of ϵ as in the definition of summable Szlenk index, [GKL] or [KOS], but one must use the differences instead of the ϵ -compressions. Thus one would begin with g and $\omega^\gamma = 1$ and let g_1 be the decreasing rearrangement of $g - 1_{(0,\epsilon_1]}$, g_2 be the decreasing rearrangement of $g_1 - 1_{(0,\epsilon_2]}$, etc. The $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ -area is zero if $g_i - 1_{(0,\epsilon_{i+1}]}$ is not non-negative for some i . This notion of summable Szlenk index seems to be the same as saying that there is a constant K such that for every $\epsilon > 0$, the ϵ area or equivalently the Szlenk index is at most $[K/\epsilon] + 1$, where $[\cdot]$ denotes the greatest integer. (See [KOS] where this latter property is called proportional index.)

Our next task is to show that there is a relation between the derivation on K and a ‘‘Szlenk’’ derivation on the probability measures on K . Below the weak*-topology on the probability measures on K is that inherited from $C(K)^*$.

Definition 3.9: Suppose that M is a set of probability measures on K and let $\delta, \epsilon > 0$. Define $M(\epsilon, \delta)^{(0)} = M$. For each $\alpha < \omega_1$, define

$$M(\epsilon, \delta)^{(\alpha)} = \{ \mu : \text{there exists } (\mu_n)_{n=1}^\infty \subset M(\epsilon, \delta)^{(\alpha)} \text{ and a sequence of closed subsets } (A_n)_{n=1}^\infty \text{ of } K \text{ such that, } \mu_n(A_n) \geq \epsilon \text{ for all } n, \text{ w}^* \lim \mu_n = \mu, d(A_n, A_m) \geq \delta \text{ for all } n \neq m \}.$$

If β is a limit ordinal, define $M(\epsilon, \delta)^{(\beta)} = \bigcap_{\alpha < \beta} M(\epsilon, \delta)^{(\alpha)}$.

Notice that the definition is at least superficially more restrictive than that of the ϵ -Szlenk subsets of M in that from the Szlenk index definition we would only have disjointness of the sets (A_n) not separation by δ . Indeed, if (μ_n) is a w^* convergent sequence of probability measures and (f_n) is a weakly null sequence of (without loss of generality) positive continuous functions such that $\int f_n d\mu_n \geq \epsilon$, then given $\epsilon' < \epsilon$, for a sufficiently small $\rho > 0$, we can let $A'_n = \{k : f_n(k) > \rho\}$ for each n and by passing to a subsequence if necessary, let $A_n = A'_n \setminus \bigcup_{k < n} A_k$, to obtain disjoint sets such that $\int_{A_n} f_n d\mu_n > \epsilon'$, for all n . Essentially this is the same as saying that the Szlenk definition detects the non-uniform absolute continuity of a set of measures. (See [A1] and the proof of Corollary 3.11.)

PROPOSITION 3.10: *Let $\epsilon, \delta > 0$. If M is a subset of the probability measures on a compact set K with δ -system of derived sets $\{K^{(\alpha)} : \alpha < \omega_1\}$ and $\mu \in M(\epsilon, \delta)^{(\alpha)}$, then for every $\epsilon' < \epsilon$, $C(\mu, \epsilon') \geq \alpha$.*

Proof: The proof is by induction on α . The main step is to prove the following.

CLAIM: *If $(\mu_n)_{n=1}^\infty \subset M$ with $C(\mu_n, \epsilon') \geq \alpha$ for each n , $w^* \lim \mu_n = \mu$, and $(A_n)_{n=1}^\infty$ is a sequence of closed subsets of K such that $\mu_n(A_n) \geq \epsilon$ and $d(A_n, A_m) \geq \delta$ for all $n \neq m$, then $C(\mu, \epsilon') \geq \alpha + 1$.*

Because the sets $K^{(\beta)}$ are closed for each β and $\mu_n \geq 0$,

$$\limsup \mu_n(K^{(\beta)}) \leq \mu(K^{(\beta)})$$

for all β . Therefore if h_n is the derived height of μ_n for each n and h is the derived height of μ , then $h(t) \geq \limsup h_n(t)$ for all t . Given $\rho > 0$ we can find $\alpha_1 < \alpha_2 < \dots < \alpha_k$ such that $\sum_{i=1}^k \mu(K^{d(\alpha_i)}) > 1 - \rho$. (Let $\alpha_0 = 0$.) Now consider $\delta_{i+1} = \limsup \mu_n((K^{(\alpha_i)} \setminus K^{(\alpha_{i+1})}) \cap A_n)$. By passing to a subsequence we may assume that this limit exists for each i and so does $\lim \mu_n(K^{(\alpha_i)})$. If $k_n \in (K^{(\alpha_i)} \setminus K^{(\alpha_{i+1})}) \cap A_n$, we know that any limit point of (k_n) is in $K^{(\alpha_{i+1})}$. Therefore, if g is a continuous function such that $1_{K^{(\alpha_{i+1})}} \leq g \leq 1$ and $\rho' > 0$,

then $(\mu_n, g) \geq \mu_n(K^{(\alpha_{i+1})}) + \mu_n(K^{(\alpha_i)} \setminus K^{(\alpha_{i+1})}) \cap A_n - \rho'$ for n sufficiently large. Consequently,

$$\begin{aligned} \mu(K^{(\alpha_{i+1})} \setminus K^{(\alpha_{i+1})}) + \mu(K^{(\alpha_{i+1})}) &= \mu(K^{(\alpha_i)}) \\ &\geq \limsup \mu_n(K^{(\alpha_{i+1})}) + \delta_{i+1}. \end{aligned}$$

Rearranging, we get

$$\mu(K^{(\alpha_{i+1})}) - \limsup \mu_n(K^{(\alpha_{i+1})}) \geq \delta_{i+1} - \mu(K^{(\alpha_{i+1})} \setminus K^{(\alpha_{i+1})}).$$

Because

$$\begin{aligned} \sum_{i=1}^k \delta_i &= \sum_{i=1}^k \limsup \mu_n((K^{(\alpha_{i-1})} \setminus K^{(\alpha_i)}) \cap A_n) \\ &\geq \limsup \mu_n(A_n \cap (K \setminus K^{(\alpha_k)})) \\ &\geq \limsup \mu_n(A_n \cap K) - \mu_n(A_n \cap K^{(\alpha_k)}) \\ &\geq \epsilon - \mu(K^{(\alpha_{k+1})}) \geq \epsilon - \rho, \end{aligned}$$

$$\sum_{i=0}^{k-1} \mu(K^{(\alpha_{i+1})}) - \lim \mu_n(K^{(\alpha_{i+1})}) \geq \sum_{i=0}^{k-1} \delta_{i+1} - \mu(K^{(\alpha_{i+1})} \setminus K^{(\alpha_{i+1})}) \geq \epsilon - 2\rho.$$

Clearly $h(t) \geq \limsup h_n(t) + 1$ for t such that $\lim \mu_n(K^{(\alpha_i)}) < t \leq \mu(K^{(\alpha_i)})$. Thus

$$\lambda(\{t : h(t) \geq \limsup h_n(t) + 1\}) \geq \epsilon - 2\rho,$$

for every $\rho > 0$. It follows there is some n such that $\lambda(\{t : h_n(t) + 1 \leq h(t)\}) > \epsilon'$. Proposition 3.5 implies that $C(h_n, \epsilon') + 1 \leq C(h, \epsilon')$, proving the Claim.

The Claim proves the induction step. Indeed, if $\mu \in M(\epsilon, \delta)^{(\alpha+1)}$ then there is a sequence $(\mu_n) \subset M(\epsilon, \delta)^{(\alpha)}$ with w^* -limit μ as in the Claim. By the inductive assumption $C(\mu_n, \epsilon') \geq \alpha$ and thus the Claim gives $C(\mu, \epsilon') \geq \alpha + 1$.

If α is a limit ordinal, let (α_n) be a sequence of ordinals converging to α . If $\mu \in M(\epsilon, \delta)^{(\alpha)}$, then $\mu \in M(\epsilon, \delta)^{(\alpha_n)}$ for all n . By the induction hypothesis $C(\mu, \epsilon') \geq \alpha_n$ for all n . Therefore $C(\mu, \epsilon') \geq \alpha$. ■

The next result is known, e.g., [S], but the apparatus we have constructed gives an easy proof.

COROLLARY 3.11: *The ϵ -Szlenk index of the unit ball of $C(\omega^\gamma \cdot k)^*$ is $\omega^\gamma[k/\epsilon] + 1$.*

Proof: We take the δ -system of derived sets to be the usual topological derived sets of $[1, \omega^\gamma \cdot k] \cup -[1, \omega^\gamma \cdot k]$ (the disjoint union of two copies of $[1, \omega^\gamma \cdot k]$), the metric to be the discrete metric $d(x, y) = 1$, for $x \neq y$, and $\delta = 1$. If μ is any

probability measure on $[1, \omega^\gamma \cdot k] \cup -[1, \omega^\gamma \cdot k]$, the derived height can be at most $\omega^\gamma \cdot k$ at each point. Thus $C(\mu, \epsilon) \leq \omega^\gamma [k/\epsilon]$.

Now consider the definition of the Szlenk subsets of the ball of $C([1, \omega^\gamma \cdot k])^*$, $P_\alpha(\epsilon')$. If $\mu \in P_{\alpha+1}(\epsilon')$ then there is a sequence of measures (μ_n) which converge w^* to μ and a sequence of norm one continuous functions (f_n) converging pointwise to 0 such that $\lim(\mu_n, f_n) \geq \epsilon'$. Let $\epsilon'' < \epsilon'$. It follows that there are disjoint sets $(A_n)_{n \in K}$ for some infinite set $K \subset \mathbb{N}$ such that $|\mu_n|(A_n) \geq \epsilon''$. Except for the absolute values this is precisely the condition in Definition 3.6. We can eliminate the absolute values by considering measures on $[1, \omega^\gamma \cdot k] \cup -[1, \omega^\gamma \cdot k]$. (Replace μ by μ' where $\mu'(A) = \mu^+(A \cap [1, \omega^\gamma \cdot k]) + \mu^-(-(A \cap -[1, \omega^\gamma \cdot k]))$.) Thus if $\mu \in P_\alpha(\epsilon')$, then $\mu \in M(1, \epsilon'')^{(\alpha)}$, where M is the set of probability measures on $\pm[1, \omega^\gamma \cdot k]$. Thus to compute the Szlenk index we may apply Proposition 3.10 to get that $\mu \in P_\alpha(\epsilon')$ implies that $C(\mu, \epsilon'') \geq \alpha$. Therefore $\alpha \leq \omega^\gamma [k/\epsilon'']$ for every $\epsilon'' < \epsilon'$, and the ϵ' Szlenk index is at most $\omega^\gamma [k/\epsilon'] + 1$. It is easy to see that $\delta_{\omega^\gamma \cdot k} \in P_{\omega^\gamma [k/\epsilon']}$, completing the proof. ■

4. The Szlenk index of the Bourgain-Delbaen space

The proof of Proposition 2.1 suggests the following approach to representing (non-uniquely) the w^* -closure of the basis $\{e_k^* : k \in \mathbb{N}\}$. Recall that a tree is a partially ordered set (T, \leq) such that each initial segment, $\{y : y \leq x\}$ for $x \in T$, is well-ordered and finite. Let $T = \bigcup_{n=0}^\infty \{0, 1\}^n$, the rooted binary tree (ordered by extension) with root the empty tuple, $()$, and let

$$W = (\{0, a, -a, b, -b, 1\} \times \{\omega \cdot m + k : m, k \in \mathbb{N} \cup \{0\}\}) \cup \{\infty\},$$

the one-point compactification of $\{0, a, -a, b, -b\} \times [1, \omega^2]$. (In this topology any sequence in W of the form $(c_i, \omega \cdot m_i + k_i)$, with $\lim m_i = \infty$, has limit ∞ .) Let K be the space of all functions from T into W in the topology of pointwise convergence. We have that K is compact by the Tychonoff theorem. Each basis vector e_k^* in X^* can be associated to a point g_k in K in the following way.

Let $g_k(()) = (1, k)$ and if $g_k(\delta_1, \delta_2, \dots, \delta_n)$ has been defined to be $(c, \omega \cdot m + \ell)$ and $\phi(\ell) = (\sigma', i, m', \sigma'', j)$, let

$$g_k(\delta_1, \delta_2, \dots, \delta_{n+1}) = \begin{cases} (\sigma' a, \omega \cdot m + i) & \text{if } \delta_{n+1} = 0, \\ (\sigma'' b, \omega \cdot \max(m, m') + j) & \text{if } \delta_{n+1} = 1. \end{cases}$$

If $\ell \leq 2$,

$$g_k(\delta_1, \delta_2, \dots, \delta_{n+1}) = (0, \omega \cdot m).$$

Define $\theta(g_k) = e_k^*$, for all k .

We need some notation to conveniently refer to the pieces of W . For $(c, \omega \cdot m + j)$ we define three functions which extract the essential parts: $V(c, \omega \cdot m + j) = c$, $Q(c, \omega \cdot m + j) = m$, and $R(c, \omega \cdot m + j) = j$. For a node \mathcal{N} of the binary tree of length $L(\mathcal{N}) = n$ and $t < n$ define the t -th truncation by $I(\mathcal{N}, t) = (\delta_1, \delta_2, \dots, \delta_t)$ if $\mathcal{N} = (\delta_1, \delta_2, \dots, \delta_n)$. We define the evaluation of an element x of X by an element $f \in K$ at a node $\mathcal{N} = (\delta_1, \delta_2, \dots, \delta_n)$ by

$$\langle f, \mathcal{N}, x \rangle = \left(\prod_{j=1}^n V(f(I(\mathcal{N}, j))) \right) \left((I - P_{Q(f(\mathcal{N}))}^*) e_{R(f(\mathcal{N}))}^* \right) x.$$

Now suppose that $x \in P_s X$ for some s and f is the preimage of e_k^* for some k , i.e., $f = g_k$. For each node $\mathcal{B} = (\delta_i)$ of T there is smallest index $n = n(\mathcal{B}, f)$ such that $R(f(I(\mathcal{B}, n))) \leq d_s$. (Of course every node with this initial segment yields the same index.)

PROPOSITION 4.1: *Let $f = g_k$, i.e., $\theta(f) = e_k^*$, for some k and $x \in P_s X$ for some s . If $\{\mathcal{N}_i\}$ is a maximal collection of incomparable nodes such that $L(\mathcal{N}_i) \leq n(\mathcal{B}, f)$ for any branch \mathcal{B} with \mathcal{N}_i as an initial segment, then the collection is finite and $e_k^*(x) = \sum_i \langle f, \mathcal{N}_i, x \rangle$.*

Proof: Observe that if \mathcal{N} is any node with $L(\mathcal{N}) < n(\mathcal{B}, f)$, $f(\mathcal{N}) = (c, \omega \cdot m + j)$ and $\phi(j) = (\sigma', r, m', \sigma'', q)$, then $j > d_s$ and

$$(4.1) \quad (I - P_m^*) e_j^*(x) = \sigma' a (I - P_m^*) e_r^*(x) + \sigma'' b (I - P_{m'}^*) (I - P_m^*) e_q^*(x).$$

Note that $(I - P_{m'}^*) (I - P_m^*) = I - P_{\max(m, m')}^*$. If $\langle f, \mathcal{N}, x \rangle = c (I - P_m^*) e_j^*(x)$, then

$$(4.2) \quad \begin{aligned} \langle f, \mathcal{N}, x \rangle &= c (\sigma' a (I - P_m^*) e_r^*(x) + \sigma'' b (I - P_{\max(m, m')}^*) e_q^*(x)) \\ &= (c \sigma' a) (I - P_m^*) e_r^*(x) + (c \sigma'' b) (I - P_{\max(m, m')}^*) e_q^*(x) \\ &= \langle f, \mathcal{N} + (0), x \rangle + \langle f, \mathcal{N} + (1), x \rangle, \end{aligned}$$

where $(\cdot) + (\cdot)$ denotes the concatenation of the tuples (\cdot) and (\cdot) . Therefore we can prove the formula by induction on the set of nodes as follows. We enumerate the nodes of the binary tree so that all nodes of a given length are labeled before any node of a longer length. Observe that the formula is obvious if we have only the node $()$ since

$$\langle f, (), x \rangle = (I - P_0^*) e_k^*(x) = e_k^*(x).$$

If this is the maximal collection, we are finished. If not, $()$ is the first node in the enumeration and we replace it by the two node collection $\{(0), (1)\}$. Formula (4.2) immediately gives the result if this is the collection of nodes. Otherwise we consider the next node in the enumeration. If it is in the collection $\{\mathcal{N}_i\}$, we move on in the enumeration; if not, we apply the formula (4.2) to replace the node by the two nodes immediately below. Note that because we began with e_k^* , with $k \leq d_r$ for some r , the integer coordinates of $\phi(k)$ are smaller than d_{r-1} . Iterating, we see that there can be only finitely many nodes in the collection $\{\mathcal{N}_i\}$. Continuing in this way we eventually reach each node in the original collection and the formula follows. ■

Our next task is to show that if $\{g_k\}$ is the set of representatives in K of the basis elements $\{e_k^*\}$ defined above, then the mapping θ described above extends to a continuous map from $\overline{\{g_k\}}$ into X^* .

Before we proceed, let us note that because of the role of $m = Q(g_k(\mathcal{N}))$, once there is a node \mathcal{N}_0 in a branch that contains 0,

$$R(g_k(\mathcal{M})) \leq Q(g_k(\mathcal{N}_0))$$

for all nodes \mathcal{M} which are descendants of \mathcal{N}_0 . Hence there can be only finitely many nodes on the branch containing \mathcal{N}_0 at which V is non-zero.

PROPOSITION 4.2: *The map θ extends to a continuous function from $\overline{\{g_k\}}$ into $\overline{\{e_k\}}$.*

Proof: Suppose that $(g_k)_{k \in M}$ has limit f in K . We have that $(g_k(\mathcal{N}))$ converges for each node \mathcal{N} . $g_k(\mathcal{N}) = (c_k, \omega \cdot m_k + j_k)$ for each k . If (m_k) is not bounded then $\lim m_k = \omega$ and limit of $(g_k(\mathcal{N}))$ is ∞ . Assume that this is not the case. Because for each k , c_k and m_k must be one of a finite set of values it follows that (c_k) and (m_k) are eventually constants c and m , respectively. If $(g_k(\mathcal{N}))$ is not eventually constant then $\lim_{k \in M} j_k = \omega$ and the limit is $(c, \omega \cdot (m + 1))$. Therefore for each node we have three possible situations.

- (1) $(g_k(\mathcal{N}))$ converges to ∞ .
- (2) $(g_k(\mathcal{N}))$ is eventually constant.
- (3) $(g_k(\mathcal{N}))$ converges to $(c, \omega \cdot (m + 1))$.

Consider in each case what happens on the nodes below.

In the first and second cases by (4.1) the same must be true for each node below \mathcal{N} . In the third case we must examine $(\phi(j_k))$ as in the proof of the Proposition 2.1. Observe that $(g_k, \mathcal{N}, x) = c_k(I - P_{m_k}^*)e_{j_k}^*(x)$ for some constant c_k and consider the same three cases. In the first case (m_k) diverges to

∞ and therefore $\lim c_k(I - P_{m_k})x = 0$ for every $x \in \bigcup_s P_s E_s$. Consequently, $w^* \lim c_k(I - P_{m_k}^*)e_{j_k}^* = 0$.

In the second case $((g_k, \mathcal{N}, x))$ is eventually constant and so is $(c_k(I - P_{m_k}^*)e_{j_k}^*)$.

In the third case (c_k) and (m_k) are eventually constant, and consequently, so is $(f, \mathcal{N}+(0), x)$.

To determine the limit of $\theta(g_k)$ we let $\{\mathcal{N}_i\}$ be the sequence of nodes such that $f(\mathcal{N}_i) \neq \infty$, $R(f(\mathcal{N}_i)) \neq 0$ and $R(f(I(\mathcal{N}_i, L(\mathcal{N}_i) - 1))) = 0$. By definition this is a set of incomparable nodes. Define $y^*(x) = \sum_i (f, \mathcal{N}_i, x)$. We claim that $w^* \lim \theta(g_k) = y^*$. Indeed the nodes we have described above are precisely the nodes corresponding to the terms that appear in the series representation for a limit point of $\theta(g_k)$ determined in the proof of Proposition 2.1. ■

Let $C = \overline{\{e_k^*\}}$.

PROPOSITION 4.3: For each $\epsilon > 0$ the ϵ -Szlenk index $\eta(\epsilon, C)$ is finite.

Proof: Fix $\epsilon > 0$. Find N such that

$$\sum_{i=N}^{\infty} a^i \sup_{m,k} \|(I - P_m)e_k^*\| < \epsilon/4.$$

Suppose that (x_k^*) is an ϵ -separated sequence in C and

$$x_k^* = \sum_{j=1}^{\infty} c_{k,j}(I - P_{m_{k,j}})e_{i_{k,j}}^*$$

and $|c_{k,j}| \leq a^j$ for all j . Then $y_k = \sum_{j=1}^{N-1} c_{k,j}(I - P_{m_{k,j}})e_{i_{k,j}}^*$, $k = 1, 2, \dots$, is an $\epsilon/2$ -separated sequence. Let (z_k) be the sequence of preimages of (x_k) corresponding to the series representation above. ($\theta(z_k) = x_k$ for all k .) Because the (y_k) is $\epsilon/2$ separated it follows that the sequence of restrictions $(z_k|_{\{\mathcal{N}: L(\mathcal{N}) < N\}})$ is distinct. Now observe that the set of maps from a finite set G into W in the topology of pointwise convergence is a metric space homeomorphic to $[1, \omega^{2 \cdot \text{card } G} (6 \cdot \text{card } G)]$. Therefore the ϵ -Szlenk index is at most $2^{N+1} + 1$. ■

COROLLARY 4.4: For each $\epsilon > 0$, $\eta(\epsilon, B_{X^*}) < \omega$. Consequently, $C(\omega^\omega)$ is not isomorphic to a quotient of X .

Proof: It is sufficient to consider $D = \overline{\text{co } \pm \{e_k^* : k \in \mathbb{N}\}}^{w^*}$ in place of B_{X^*} . By the Choquet theorem we can associate each element x^* of D to some probability measure μ_{x^*} on

$$C = \overline{\pm \{e_k^* : k \in \mathbb{N}\}}^{w^*}.$$

Observe that if (x_n^*) is a w^* -convergent sequence in D and (x_n) is a weakly null sequence in the unit ball of X such that $\lim x_n^*(x_n) \geq \epsilon_1$, then there exist an infinite subset L of \mathbb{N} and $\epsilon/4$ norm separated subsets $(A_n)_{n \in L}$ of C such that $\mu_{x_n^*}(A_n) \geq \epsilon_1/2$ for all $n \in L$.

Now we consider the modified Szlenk subsets of C , $\{P_\alpha(\epsilon_1/4, C) : \alpha < \omega_1\}$, as the δ -system of derived sets with $\delta = \epsilon_1/4$ and let

$$M = \{\mu : \mu \text{ is a probability measure representing some } x^* \in D\},$$

$\epsilon = \epsilon_1/2$ and $\delta = \epsilon_1/4$. By Proposition 3.10, if $\mu \in M(\epsilon, \delta)^{(\alpha)}$ then $C(\mu, \epsilon_1/4) \geq \alpha$. However, $C(\mu, \epsilon_1/4)$ must be finite by Proposition 4.3. ■

Remark 4.5: From the proofs of Proposition 4.3 and the corollary, the ϵ -Szlenk index can be estimated from above. Recently Haydon [Ha] has shown that the Bourgain–Delbaen spaces are hereditarily ℓ_p for some p which depends on a and b . From this one can get a lower estimate on the Szlenk index. Using results in [GKL, GKL1] it follows that these spaces are not uniformly homeomorphic to c_0 .

Remark 4.6: After reading an earlier version of this paper I. Gasparis communicated to us another method of showing that the ϵ -Szlenk index of the Bourgain–Delbaen space is finite without determining the behavior of the index. With his permission we include a sketch of the argument here.

$\eta(\epsilon, B_{X^*}) \geq \omega$ for some $\epsilon > 0$ is equivalent to the statement that $C(\omega^\omega)$ is a quotient of X . (See [AB].) It is well-known that $C(\omega^\omega)$ has ℓ_1 as a spreading model of a weakly null sequence. If $C(\omega^\omega)$ is a quotient of X , then X also has a weakly null sequence with spreading model ℓ_1 . This would imply that the basis of X has blocks that are equivalent to the basis of ℓ_1^n for all n . However, the proof of Lemma 5.3 of [BD] or Proposition 3.9 of [B] shows that this is impossible.

5. Final remarks

The arguments given above suggest that there is considerable flexibility in the construction given by Bourgain and Delbaen. One possibility is to replace the binary nature of the construction by one which allows a greater number of terms. Thus in place of $(\pm a, \pm b)$ one might have a collection of finite sequences $(a_n^j)_{n=1}^N, j = 1, 2, \dots, J$. Then the new functionals might evaluate as $\sum_{n=1}^N a_n^k e_{s_n}^*(i_n \pi_n x - i_{n-1} \pi_{n-1} x)$, where (s_n) is a sequence such that $d_{n-1} < s_n \leq d_n$ for each n and d_n is the cardinality of the set of coordinates defined by the n -th stage of the construction. Some care would need to be taken to preserve the boundedness of the iterated embeddings. It would be most interesting if the set

of finite sequences could be made to vary and if the sequence of finite segments of the integers could be replaced by finite branches of a tree. This might be an approach to answering the following question.

QUESTION 5.1: *Given a countable ordinal α , is there a \mathcal{L}_∞ -space X_α such that X_α does not contain c_0 and X_α has Szlenk index ω^α ?*

One other observation is that much of what we have done still works if $a = 1$. What does not work is the argument in Proposition 2.1 to find the convergent series for each element of the dual. Thus the corresponding set K is more complicated and seems to include a Cantor set of well separated points. A thorough analysis of this case might yield some additional information about the first example in [BD]. Finally, note that we have not used the extra conditions imposed on a and b in [BD] to get a somewhat (hereditarily) reflexive example.

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